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GEOMETRY: EXAMPLES 1

1. Given a real constant c, let $\Sigma = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = c\}$.
   (a) Sketch Σ for a representative selection of values of c.
   (b) For which values is Σ an embedded surface in $\mathbb{R}^3$? Justify your answer.
   (c) Write down parametrisations covering Σ in the cases where it *is* an embedded surface.

2. Show that a subset $\Sigma \subset \mathbb{R}^3$ is an embedded surface iff for all p in Σ there exists an open neighbourhood T of p in $\mathbb{R}^3$, an open neighbourhood W of the origin in $\mathbb{R}^3$, and a diffeomorphism $g : T \to W$ such that $g(T \cap \Sigma) = W \cap (\mathbb{R}^2 \times \{0\})$.

3. Show that the map $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$ defined by
$$\sigma(u, v) = \Big(\frac{u}{1 + u^4}, v, \frac{u^2}{1 + u^4}\Big)$$
is smooth, that it is a bijection onto its image Σ, and that $D_q\sigma$ is injective for all $q \in \mathbb{R}^2$. Sketch Σ and show that it is not an embedded surface.

4. Consider the polar and Cartesian parametrisations $\sigma(r, \theta) = (r \cos\theta, r \sin\theta)$ and $\tau(x, y) = (x, y)$ of $\mathbb{R}^2$ near $p \neq 0$.
   (a) Sketch the vectors σ_r, σ_θ, τ_x, and τ_y at a selection of points.
   (b) Using the the change of basis formula from lectures, express τ_x and τ_y in terms of σ_r and σ_θ. [*Hint: use the chain rule to avoid expressing r and θ in terms of x and y.*]
   (c) Verify your answer using explicit expressions for σ_r and σ_θ.

5. Let $F : \Sigma_1 \to \Sigma_2$ be a smooth map between embedded surfaces. Given $p \in \Sigma_1$ define a map
$$D_pF : T_p\Sigma_1 \to T_{F(p)}\Sigma_2$$
as follows. Pick parametrisations σ_1 of Σ_1 near p and σ_2 of Σ_2 near $F(p)$ with WLOG $\sigma_1(0) = p$ and $\sigma_2(0) = F(p)$. Note that $D_0\sigma_1$ gives a linear isomorphism $\mathbb{R}^2 \to T_p\Sigma_1$. Define
$$D_pF = D_0\sigma_2 \circ D_0(\sigma_2^{-1} \circ F \circ \sigma_1) \circ (D_0\sigma_1)^{-1}.$$
   (a) Show that this is independent of the choices of σ_1 and σ_2.
   (b) Show that if $v \in T_p\Sigma_1$ is given by $\dot\gamma(0)$, for a smooth map $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ with image contained in Σ_1 and with $\gamma(0) = p$, then $D_pF(v) = (F \circ \gamma)^\bullet(0)$. [*Hint: Write γ in the form $\sigma_1 \circ \Gamma$.*]

6. Let $\sigma^\pm : \mathbb{R}^2 \to S^2$ be the inverse stereographic projection parametrisations, given by
$$\sigma^\pm(u, v) = \frac{1}{u^2 + v^2 + 1}\big(2u, 2v, \pm(u^2 + v^2 - 1)\big).$$
Identify $\mathbb{R}^2$ with $\mathbb{C}$ and let $P : \mathbb{C} \to \mathbb{C}$ be a non constant complex polynomial. Define $F : S^2 \to S^2$ by
$$F(p) = \begin{cases} \sigma^+ \circ P \circ (\sigma^+)^{-1}(p) & \text{if } p \in S^2 \setminus \{(0, 0, 1)\} \\ (0, 0, 1) & \text{if } p = (0, 0, 1). \end{cases}$$
   (a) Show that F is smooth.
   (b) For $P(\zeta) = \zeta^3 + \zeta^2 + 1$, at which points is F a local diffeomorphism?

7. Show that the Möbius band, defined to be the image of the map $\sigma : (-1, 1) \times \mathbb{R} \to \mathbb{R}^3$ given by
$$\sigma(u, v) = ((2 + u \cos\tfrac{v}{2}) \cos v, (2 + u \cos\tfrac{v}{2}) \sin v, u \sin\tfrac{v}{2}),$$
is not orientable. [*Hint: Take Gauss maps n_1 and n_2 defined where $v \in (-\pi, \pi)$ and $v \in (0, 2\pi)$ respectively, and compare a putative global Gauss map n with them.*]

8. Suppose that Σ is an embedded surface defined by the vanishing of single function, i.e. that $\Sigma = f^{-1}(0)$ where f is a smooth function on an open subset of $\mathbb{R}^3$ satisfying $D_pf \neq 0$ for all $p \in \Sigma$. Show that Σ is orientable, and deduce that there exist embedded surfaces that cannot be defined by the vanishing of a single function.